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Rational homotopy type of projectivization of the tangent bundle of certain spaces

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Abstract

Purpose – The paper aims to determine the rational homotopy type of the total space of projectivized bundles over complex projective spaces using Sullivan minimal models, providing insights into the algebraic structure of these spaces.

Design/methodology/approach – The paper utilises techniques from Sullivan's theory of minimal models to analyse the differential graded algebraic structure of projectivized bundles. It employs algebraic methods to compute the Sullivan minimal model of $P(E)$ and establish relationships with the base space.

Findings – The paper determines the rational homotopy type of projectivized bundles over complex projective spaces. Of great interest is how the Chern classes of the fibre space and base space, play a critical role in determining the Sullivan model of *P(E)*. We also provide the homogeneous space of *P(E)* when $n = 2$. Finally, we prove the formality of *P(E)* over a homogeneous space of equal rank.

Research limitations/implications – Limitations may include the complexity of computing minimal models for higher-dimensional bundles.

Practical implications – Understanding the rational homotopy type of projectivized bundles facilitates computations in algebraic topology and differential geometry, potentially aiding in applications such as topological data analysis and geometric modelling.

Social implications – While the direct social impact may be indirect, advancements in algebraic topology contribute to broader mathematical knowledge, which can underpin developments in science, engineering, and technology with societal benefits.

Originality/value – The paper's originality lies in its application of Sullivan minimal models to determine the rational homotopy type of projectivized bundles over complex projective spaces, offering valuable insights into the algebraic structure of these spaces and their associated complex vector bundles.

Keywords Projectivization bundle, Sullivan minimal models, Formality, Complex manifolds

Paper type Research paper

1. Introduction

This section will outline the fundamental concepts and definitions of differential graded algebras. We consider a setting where all algebras and vector spaces are taken over the field $\mathbb{Q}$ of rational numbers. The primary reference for the definitions in this paper is [1].

Definition 1.1. A graded algebra is a graded vector space $A = \oplus_{p \geq 0} A^p$ together with an associative multiplication of degree zero:

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$$A^p \otimes A^q \to A^{p+q}, \qquad x \otimes y \to xy,$$

such that there is an identity $1 \in A^0$. A graded algebra is called commutative (cga) if

$$xy = (-1)^{|x||y|} yx,$$

where, x, y are homogeneous elements of degree $|x|$ and $|y|$ respectively. A differential in a graded algebra A is a linear map $d: A^p \to A^{p+1}$ satisfying $d \circ d = 0$,

$$d(xy) = (dx)y + (-1)^{|x|} x(dy).$$

The pair (A, d) is a commutative differential graded algebra (cdga), if A is commutative.

If V is a graded vector space, then the free commutative graded algebra ΛV is defined by $\Lambda V = S(V^{even}) \otimes E(V^{odd})$, where $S(V^{even})$ is the symmetric algebra and $E(V^{odd})$ is the exterior algebra. If $\{v_1, v_2, \cdots\}$ is a basis of V, then ΛV is often written as $\Lambda(v_1, v_2, \cdots)$.

Definition 1.2. Let $C^*(X, \mathbb{Q})$ be the cochain algebra of the normalized singular cochains on a topological space X. Sullivan defines a cdga $A_{PL}(X, \mathbb{Q})$ of polynomial forms on X, with natural cochain algebra quasi-isomorphisms

$$C^*(X, \mathbb{Q}) \xrightarrow{\cong} D(X) \xleftarrow{\cong} A_{PL}(X, \mathbb{Q}),$$

where $D(X)$ is a third natural cochain algebra [2, §10]. Moreover, $A_{PL}(X, \mathbb{Q})$ is a contravariant functor from the category of topological spaces to a category of cdgas.

Definition 1.3. A commutative cochain algebra $(\Lambda V, d)$ is called a Sullivan algebra if

$$V = \bigcup_{k=0}^{\infty} V(k)$$

such that, $d = 0$ in $V(0)$ and d: $V(k) = \Lambda V(k-1)$, $k \geq 1$. Moreover, a Sullivan algebra $(\Lambda V, d)$ is called minimal if $dV \subset \Lambda^{\geq 2} V$. Let (A, d) be a cdga with $H^0(A) = \mathbb{Q}$, there always exists a quasi-isomorphism $m: (\Lambda V, d) \to (A, d)$, where $(\Lambda V, d)$ is a Sullivan algebra. A cdga map $\varphi: (A, d) \to (B, d)$ is a quasi-isomorphism if $H^*(\varphi)$ is an isomorphism.

Definition 1.4. Consider the cdga $\Lambda(t, dt)$, where $|t| = 0$, $|dt| = 1$ and $d(t) = dt$. There are augmentation maps $\varepsilon_i: \Lambda(t, dt) \to \mathbb{Q}$, where $\varepsilon_0(t) = 0$ and $\varepsilon_1(t) = 1$. Two cdga maps ϕ_0, ϕ_1: $(\Lambda V, d) \to (B, d)$ are homotopic (i.e. $\phi_0 \simeq \phi_1$) if there exists a cdga map $\Phi: (\Lambda V, d) \to B \otimes \Lambda(t, dt)$ such that $(1 \otimes \varepsilon_i) \circ \Phi = \phi_i$ [2, §12].

Definition 1.5. [1, §2] Let X be a path-connected space. The Sullivan minimal model of X is the Sullivan minimal model of $A_{PL}(X)$. If $f: X \to Y$ is a map between path-connected spaces, the minimal model of $A_{PL}(f)$ is called the Sullivan minimal model of f.

Let $\varphi: (A, d) \to (B, d)$ be a map of cdga's, and $m_A: (\Lambda V, d) \to (A, d)$ and $m_B: (\Lambda W, d) \to (B, d)$ be the Sullivan models. Then there exists a morphism of cdga's $g: (\Lambda V, d) \to (\Lambda W, d)$ that is unique up to homotopy, such that $m_B \circ g \simeq \varphi \circ m_A$ is called the Sullivan minimal model of φ.

Definition 1.6. A Sullivan minimal algebra $(\Lambda V, d)$ is said to be formal if there exists a homomorphism $\phi: (\Lambda V, d) \to H*(\Lambda V, d)$ inducing an isomorphism on cohomology. A space X is said to be formal if its minimal model is formal.

Definition 1.7. A relative Sullivan model of a morphism of commutative differential graded algebras $\varphi: (A, d) \to (B, d)$ is a morphism $(A, d) \xrightarrow{i} (A \otimes \Lambda V, d)$ and $V = \cup_{k \geq 0} V(k)$, where $V(0) \subset V(1) \subset \cdots$ is an increasing sequence with $d(V(0)) \subseteq A$ and $dV(k) \subseteq A \otimes V(k-1)$, $k \geq 1$ and there is a quasi isomorphism $\psi: (A \otimes \Lambda V, d) \to (B, d)$ such that $\psi \circ i = \varphi$.

Let $F \to E \xrightarrow{\xi} B$ be a fibration between simply connected spaces, then there exists a relative Sullivan model $(\Lambda V, d) \to (\Lambda V \otimes \Lambda W, D) \to (\Lambda W, d)$, where $(\Lambda V, d)$ and $(\Lambda W, d)$ are respective Sullivan models of B and F. Moreover, $(\Lambda V \otimes \Lambda W, D)$ is a Sullivan model of E, not necessarily minimal [3, §12].

Definition 1.8. Let Q be a finite-dimensional graded vector space concentrated in even degrees. A regular sequence is defined as an ordered set of elements $u_1, \ldots, u_m$ belonging to $\Lambda^+ Q$ such that u_1 is not a zero divisor in ΛQ, and for $i \geq 2$, then the image of u_i is likewise not a zero divisor in the quotient graded algebra $\Lambda Q/(u_1, \ldots, u_{i-1})$. In particular, any given sequence of the form $u_1, \ldots, u_m$ can be used to define a pure Sullivan algebra denoted as $(\Lambda Q \otimes \Lambda P, d)$, for $\Lambda P = \Lambda(x_1, \ldots, x_m)$, where the differential operator is defined by $dx_i = u_i$ [2, p. 437, 4 p. 157].

Definition 1.9. [1, p. 188] A closed manifold (M^{2n}, ω) is cohomologically symplectic (or c-symplectic) if there is $\omega \in H^2(M; \mathbb{Q})$ such that $\omega^n \neq 0$.

2. Model of the projectivization of a complex bundle

A projectivized bundle is constructed by replacing each fibre of a complex vecto bundle with the corresponding projective space. Specifically, let

$$\pi : \mathbb{C}^n \to E \to B,$$

be a complex vector bundle over a complex smooth manifold B, the projectivized bundle $P(E)$ is constructed by replacing each fibre $\mathbb{C}^n$ of E with the corresponding projective space $\mathbb{C}P^{n-1}$ [5, §20]. The operation of projectivization applied to the bundle π yields a fibre bundle,

$$P(\pi): \quad \mathbb{C}P^{n-1} \to P(E) \to B. \tag{1}$$

The cohomology algebra of the total space $P(E)$ is given by

$$H^*(P(E)) = H^*(B)[x]/Bx\left(x^n + c_1 E x^{n-1} + \cdots + c_{n-1} E x + c_n E\right),$$
$$\times \left(x^n + c_1(E)x^{n-1} + \cdots + c_{n-1}(E)x + c_n(E)\right),$$

where $c_i \in H(B, \mathbb{Q})$ are the Chern classes on the complex bundle π and x is a generator of $H^*(\mathbb{C}P^{n-1}; \mathbb{Q})$ [1, p. 333, 4, §4, 6, p. 1963]. According to the Leray-Hirsch theorem, $H^*(P(E)) \cong H^*(B) \otimes H^*(\mathbb{C}P^{n-1})$ as vector spaces [5].

Theorem 2.1. *Let $\pi: \mathbb{C}^n \to E \to B$ be a complex fibre bundle and $P(\pi)$ its projectivization. If (A, d) is a cdga model of B, then a model of the total space of $P(\pi)$ is given by $(A \otimes \Lambda(x_{2n}, x_{2n-1}), D)$, $Dx_{2n-1} = x_2^n + \sum_{i=1}^n c_i x_2^{n-i}$ and $c_i \in H^{2i}(A, \mathbb{Q})$ represent the Chern classes of π.*

Proof

If B is a complex manifold and π corresponds to the tangent bundle, the structure group of the complex vector bundle can be reduced to $\mathrm{U}(n)$. Moreover, the structural group of $P(\pi)$ reduces to $\mathrm{U}(n)/S^1 \cong P\mathrm{U}(n)$, where S^1 is considered as a subgroup of $\mathrm{U}(n)$ under the identification

$$\lambda \mapsto \begin{pmatrix} \lambda & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda \end{pmatrix}, \quad \lambda \in S^1.$$

Therefore, $P(\pi)$ is classified by a map

$$f : B \to BP\mathrm{U}(n).$$

As $BP\mathrm{U}(n)$ has the rational homotopy type $S\mathrm{U}(n)$, a Sullivan model of $BP\mathrm{U}(n)$ is given by $(\Lambda(y_4, \cdots y_{2n}), 0)$, and a model of f is ϕ: $(\Lambda(y_4, \cdots y_{2n}), 0) \to (A, d)$ with Chern classes $[c_i] = \phi(y_{2i}) \in H^{2i}(A, d)$, for $i = \{1, 2, \ldots, n\}$. A relative model of projectivization is then given by,

$$\Phi : (A, d) \to (A \otimes (\Lambda(x_2, x_{2n-1}), D) \to (\Lambda(x_{2n}, x_{2n-1}), d),$$

with

$$Dx_{2n-1} = x_2^n + \sum_{i=1}^{n} c_i x_2^{n-i}.$$

This agrees with the rational homotopy type of the rational universal fibration of fibre $\mathbb{C}P^{n-1}$ [7, 8, §11].

3. Projectivization and tangent sphere bundles over complex manifolds

For a complex vector bundle $\pi: E \to B^{2n}$ let the unit tangent sphere bundle be denoted by $S^{2n-1} \to S(E) \to B^{2n}$. Also, the complex structure on E implies that the circle S^1 acts on the sphere bundle. Therefore, there exists a bundle map ζ: $S(E) \to P(E)$.

$$\begin{array}{ccccc} S^1 & \hookrightarrow & S^{2n-1} & \longrightarrow & \mathbb{C}P^{n-1} \\ & & \downarrow & & \downarrow \\ S^1 & \hookrightarrow & S(E) & \xrightarrow{\ \zeta\ } & P(E) \\ & & \downarrow & & \downarrow \\ & & B^{2n} & = & B^{2n} \end{array}$$

A model for the unit sphere bundle is given by $(A \otimes \Lambda v_{2n-1}, D)$, where $D_{|A} = d, Dv_{2n-1} = w$, where w is a cocycle that represents the fundamental class of B [9]. The following diagram of cdga's commutes:

$$
\begin{array}{ccccc}
(A,d) & \longrightarrow & (A\otimes\Lambda(x_2,x_{2n-1}),D) & \longrightarrow & (\Lambda(x_2,x_{2n-1}),d) \\
\Big\| & & \Big\downarrow q & & \Big\downarrow \overline{q} \\
(A,d) & \longrightarrow & (A\otimes\Lambda(v_{2n-1}),D) & \longrightarrow & (\Lambda v_{2n-1},0)
\end{array}
$$

where, $q(x_2) = 0$ and $q(x_{2n-1}) = v_{2n-1}$.

Theorem 3.1. *Consider the fibration of the unit tangent sphere bundle over a complex projective space $\mathbb{C}P^n$ and the projectivization fibration over $\mathbb{C}P^n$,*

$$S(E)\xrightarrow{\zeta}P(E),$$

then $\pi_i(\zeta)\otimes\mathbb{Q}$ yields $\pi_i(P(E))\otimes\mathbb{Q}\simeq\pi_i(S(E))\otimes\mathbb{Q}$, for i ≥ 3.

Proof: If $\mathbb{C}^n\to E\to\mathbb{C}P^n$ is a tanget bundle and y_2 a generater of $H^2(\mathbb{C}P^n,\mathbb{Q})$, then the Chern classes are given by $y_2^i\in H^{2i}(\mathbb{C}P^n,\mathbb{Q})$, $i=1,2,\ldots,n$ up to a non-zero rational factor (see [4, §21]). Given that the Sullivan model of $P(E)$ is $(\Lambda(y_2,\ y_{2n+1})\otimes\Lambda(x_2,\ x_{2n-1}),\ d)$ with $dy_2=dx_2=0$, $dy_{2n+1}=y_2^{n+1}$, and $dx_{2n-1}=x_2^n+\cdots+x_2^2y_2^{n-2}+y_2^n$. Also a model of the sphere bundle $S(E)$ is given by the Sullivan model $(\Lambda(y_2,\ y_{2n+1})\otimes\Lambda v_{2n-1},\ d)$, with $dy_2=0$, $dy_{2n+1}=y_2^{n+1}$ and $dv_{2n-1}=y_2^n$. The Sullivan model of ζ is

$$q:(\Lambda(y_2,y_{2n+1})\otimes\Lambda(x_2,x_{2n-1}),d)\to(\Lambda(y_2,y_{2n+1})\otimes\Lambda v_{2n-1},D),$$

where $q(x_2)=0$, $q(x_{2n-1})=v_{2n-1}$ (see Ref. [10]).

The dual homotopy groups generated by $[\pi_*(P(E))\otimes\mathbb{Q}]^\#\cong\langle y_2,y_{2n+1},x_2,x_{2n-1}\rangle$ and $[\pi_*(S(E))\otimes\mathbb{Q}]^\#\cong\langle y_2,y_{2n+1},v_{2n-1}\rangle$. The restriction of q to the algebra generators yields

$$\overline{q}:\langle y_2,y_{2n+1},x_2,x_{2n-1}\rangle\to\langle y_2,y_{2n+1},v_{2n-1}\rangle$$

where $\overline{q}(y_2)=y_2$, $\overline{q}(y_{2n+1})=y_{2n+1}$, $\overline{q}(x_2)=0$, and $\overline{q}(x_{2n-1})=v_{2n-1}$. Hence, $\overline{q}$ is surjective and hence $\pi_*(\zeta)\otimes\mathbb{Q}$ is injective. Moreover, $\overline{q}$ is an isomorphism in the degree greater than 2. Hence, for $i\geq 3$, $\pi_i(\zeta)\otimes\mathbb{Q}$ is an isomorphism.

Theorem 3.2. The rational homotopy type of the total space P(E) of the projectivized bundle

$$\mathbb{C}P^1\to P(E)\to\mathbb{C}P^2,$$

is that of the homogeneous space U(3)/U(1) × U(1) × U(1).

Proof: Consider the projectivization fibration $\mathbb{C}P^1\to P(E)\to\mathbb{C}P^2$. The Sullivan model of the total space $P(E)$ is given by

$$(\Lambda(y_2,y_5)\otimes\Lambda(x_2,x_3),d),$$

with $dy_2=dx_2=0$, $dx_3=x_2^2+x_2y_2+y_2^2$ and $dy_5=y_2^3$ which is quasi isomorphic to $((\Lambda y_2/y_2^3)\otimes\Lambda(x_2,x_3);\overline{d})$, $\overline{d}y_2=\overline{d}x_2=0$ and $\overline{d}x_3=x_2^2+x_2y_2+y_2^2$. The cohomology ring is

$$H^*(P(E),\mathbb{Q}) = \left(\Lambda y_2/y_2^3\right) \otimes \left(\Lambda x_2/\left(x_2^2 + x_2 y_2 + y_2^2\right).\right).$$

Now, consider the homogeneous space G/H where $G = \mathrm{U}(3)$ and $H = \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$.

Let $j: H = \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1) \hookrightarrow G = \mathrm{U}(3)$ the inclusion, and $B_j: BH \to BG$ the classifying map. Then G/H is the homotopy pullback of the following diagram:

$$\begin{array}{ccc} G/H & \longrightarrow & EG \\ \downarrow & & \downarrow \\ BH & \xrightarrow{B_j} & BG \end{array}$$

A Sullivan model for G/H is

$$(\Lambda(a_2, b_2, c_2) \otimes \Lambda(w_1, w_3, w_5), d),$$

where

$$\begin{aligned} dw_3 &= a_2 b_2 + b_2 c_2 + a_2 c_2, \\ dw_5 &= a_2 b_2 c_2, \\ dw_1 &= a_2 + b_2 + c_2. \end{aligned}$$

A change of variable $t_2 = a_2 + b_2 + c_2$ yields an isomorphism $(\Lambda(a_2, b_2, t_2) \otimes \Lambda(w_1, w_3, w_5), d)$ with $dw_1 = t_2$, $dw_3 = a_2 b_2 + (a_2 + b_2)(t_2 - a_2 - b_2)$ and $dw_5 = a_2 b_2 (t_2 - a_2 - b_2)$ which is quasi-isomorphic to

$$(\Lambda(a_2, b_2, w_3, w_5), d),$$

where

$$\begin{aligned} da_2 &= db_2 = 0, \\ dw_3 &= -a_2^2 - a_2 b_2 - b_2^2, \\ dw_5 &= -a_2^2 b_2 - a_2 b_2^2. \end{aligned}$$

The map $f: (\Lambda(a_2, b_2, w_3, w_5), d) \to (\Lambda(x_2, y_2, x_3, y_5), d)$, with $f(a_2) = x_2$, $f(w_3) = x_3$, $f(b_2) = y_2$, $f(w_5) = y_2 x_3 - y_5$ is a quasi-isomorphism. Therefore, $P(E)$ has the rational homotopy type of $\mathrm{U}(3)/\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$.

Proposition 3.3.

Consider a non-trivial complex vector bundle $\delta: \mathbb{C}^n \to E \to S^{2n}$ and the projectivization of vector bundle

$$P(\delta): \mathbb{C}P^{n-1} \to P(E) \xrightarrow{\pi} S^{2n}.$$

Then $P(E)$ has a rational homotopy type of $\mathbb{C}P^{2n-1}$, for $n \geq 2$.

Proof:
The Sullivan minimal model of S^{2n} is $(\Lambda(a_{2n}, b_{4n-1}), d)$ with $da_2 = 0$, and $db_{4n-1} = a_{2n}^2$. Again, the Sullivan minimal model of $\mathbb{C}P^{n-1}$ is $\Lambda(x_2, x_{2n-1})$ with $dx_2 = 0$, and $dx_{2n-1} = x_2^n$. Therefore, the KS model of π;

$$(\Lambda(a_{2n}, b_{4n-1}), \mathrm{d}) \to (\Lambda(a_{2n}, b_{4n-1}) \otimes \Lambda(x_2, x_{2n-1}), D) \to (\Lambda(x_2, x_{2n-1}), d),$$

is classified by a mapping,

$$f : (\Lambda(y_4, \ldots, y_{2n}), 0) \to (\Lambda(a_{2n}, b_{4n-1}), \mathrm{d}).$$

The Chern classes given by $c_1 = [f(y_2)] = 0$, $c_2 = [f(y_4)] = 0$, $c_3 = [f(y_6)] = 0, \cdots,$ $c_{n-1} = [f(y_{2(n-1)})] = 0$, $c_n = [f(y_{2n})] = [a_{2n}] \in H*(S^{2n})$.

Then, the total space $P(E)$ of the projectivized bundle has a Sullivan model,

$$(\Lambda(a_{2n}, b_{4n-1}, x_2, x_{2n-1}), D),$$
$$Da_{2n} = 0,\ Dx_2 = 0,\ Db_{4n-1} = a_{2n}^2,\ Dx_{2n-1} = x_2^n + a_{2n}.$$

Making the change of variables to eliminate the linear part, let $t_{2n} = x_2^n + a_{2n}$. Then we get an isomorphic cdga

$$(\Lambda(t_{2n}, b_{4n-1}, x_2, x_{2n-1}), D),$$
$$Da_{2n} = 0,\ Dx_2 = 0,\ Db_{4n-1} = \left(t_{2n} - x_2^n\right)^2,\ Dx_{2n-1} = t_{2n}.$$

As the ideal (x_{2n-1}, t_{2n}) is acyclic, the minimal Sullivan model of $P(E)$ is,

$$(\Lambda(x_2, b_{4n-1}), D),$$
$$Dx_2 = 0,\ Db_{4n-1} = x_2^{2n},$$

Hence $P(E)$, has a rational homotopy type of $\mathbb{C}P^{2n-1}$.
Remark: As $P(E) \cong_{\mathbb{Q}} \mathbb{C}P^{2n-1}$, then $P(E)$ satisfies the hard Lefschetz property (See Theorem 3.1 in Ref. [6]).

Theorem 3.4. ([4, p. 149, 6], **Theorem 4.1**). *Let M be a simply connected smooth manifold of dimension $2n$. Given a fibration*

$$\mathbb{C}P^{n-1} \to E \to M,$$

then E is formal if and only if M is formal.

The proof of this Theorem 3.4 is given in Refs. [4, 11]. We give here a simple proof of a particular case of this theorem.

Theorem 3.5. *If B is a homogenous space of equal rank with a complex structure of dimension n, and $\mathbb{C}^n \to E \to B$ is the complex vector bundle, then in the projectivization bundle*

$$\mathbb{C}P^{n-1} \to P(E) \to B,$$

have a formal total space $P(E)$.

Proof:
Let B be a homogeneous space of equal rank, implying the existence of a pure model

$$(\Lambda(y_1, \ldots, y_m, v_1, \ldots, v_m),\ d),$$

where y_i and v_i are even and odd generators respectively, $dy_i = 0$, and $dv_i \subseteq \Lambda(y_1, \ldots, y_m)$. Moreover, $(dv_1, \ldots, dv_m)$ is a regular sequence in $\Lambda(y_1, \ldots, y_m)$. Hence B is formal. A model for the total space of the projectivized bundle is given by

$$(\Lambda(y_1, \ldots, y_m, x_2, v_1, \ldots, v_m, x_{2n-1}),\ d),$$

where

$$dy_1 = \cdots = dy_m = dx_2 = 0 \ \text{ and } \ dx_{2n-1} = x_2^n + \sum_{i=1}^{n} c_i x_2^{n-i}, \quad [c_i] = H^{2i}(B).$$

Let $u_i = dv_i$ we show that $(u_1, \ldots, u_m, dx_{2n-1})$ forms a regular sequence in $\Lambda(y_1, \ldots, y_m, x_2)$. It suffices to show that

$$dx_{2n-1} = x_2^n + \sum_{i=1}^{n} c_i x_2^{n-i}$$

is not a zero divisor in $(\Lambda(y_1, \ldots, y_m)/(u_1, \ldots, u_m)) \otimes \Lambda x_2)$. By the contrary, assume that

$$x_2^n + \sum_{i=1}^{n} c_i x_2^{n-i}$$

is a zero divisor. Then there exists

$$\beta_0 + \sum_{k=1}^{m} \beta_k x_2^k,$$

such that

$$\left(\beta_0 + \sum_{k=1}^{m} \beta_k x_2^k\right)\left(x_2^n + \sum_{i=1}^{n} c_i x_2^{n-i}\right) = 0, \tag{2}$$

where $\beta_i \in \Lambda(y_1, \ldots, y_m)/(u_1, \ldots, u_m)$. The expansion of equation (2) yields

$$\beta_m x_2^{m+n} + \text{polynomial of degree} < m+n \ \text{ in } \ x_2.$$

This cannot be a zero divisor in $(\Lambda(y_1, \ldots, y_m)/(u_1, \ldots, u_m))$.

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