GWANDA STATE UNIVERSITY

FACULTY OF COMPUTATIONAL SCIENCES

DEPARTMENTS OF MATHEMATICS AND STATISTICS

COMPLEX ANALYSIS

CMS 2206

Examination Paper

April 2025

Time Allowed:	**3 hours**
Total Marks:	**100**
Examiner's Name:	**Mr. M. Mpofu**

INSTRUCTIONS

Candidates should answer **ALL** questions in Section A and attempt **ANY THREE** questions in Section B.

ADDITIONAL REQUIREMENTS

Scientific calculator

SECTION A (40 marks)

Answer ALL questions from this section.

A1. Evaluate,

$$I = \oint_{|z|=3} \frac{z^2}{z^2+1}$$

[5]

A2. Given that Cauchy-Riemann equations in cartesian coordinates as

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Show that the Cauchy-Riemann equations in polar coordinates are given by

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

where $x = r\cos\theta$ and $y = r\sin\theta$. [10]

A3. By applying the Jordan's Lemma evaluate

$$\int_{-\infty}^{\infty} \frac{x \sin 5x}{x^2+4} dx$$

[5]

A4. (i) Determine the convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}+3}(4-i)^n$$

[5]

(ii) Find the radius and centre of convergence for the series

$$\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}(z-3i)$$

[7]

A5. Prove the extended Cauchy's integral formula. [8]

SECTION B (60 marks) Answer ANY TWO questions from this section.

B6. (i) Graphically represent the values of z where

$$\left|\frac{z-5}{z+5}\right| = \frac{3}{2}$$

[3]

(ii) Given that $u(x,y) = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. Find the conjugate of v and express $f(z) = u + iv$ as a function of z. [5]

(iii) Evaluate the following

(a)

$$I = \oint_{\left|z+\frac{3}{2}\right|=1} \frac{e^z}{z^2+2z} dz$$

[3]

(b)

$$I = \oint_{|z|=2} \frac{\sin iz}{z^2-4z+3} dz$$

[5]

(iv) Use the extended Cauchy's integral to evaluate

$$I = \oint_{\left|z-\frac{1}{2}\right|=1} \frac{\cosh z}{(z+1)^3(z-1)} dz$$

[4]

B7. (i) (a) State the ML Theorem [2]

(b) Use the ML Theorem to bound the function:

$$I = \oint_c \frac{dz}{z(z-2)^3}$$

where c is a circle, centre origin and radius 5. [5]

(ii) Find the Laurent series of

$$f(z) = \frac{e^z}{z}, \quad \text{about} \quad z = 0$$

(a) using the formula for An [5]

(b) using the Taylor series for e^z [5]

(iii) Find the analytic function $w = f(z)$ if its imaginary part is $v(x,y) = 3x + 2xy$ and if $f(-i) = 2$. [3]

B8. (i) Define

(a) Essential singularity. [2]

(b) Removable singularity. [2]

(ii) Prove that if $f(z)$ has an nth-order pole at $z = a$, then

$$Res(f(a)) = \frac{1}{(n-1)} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z)\right]$$

[6]

(iii) Show that

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^4} = \frac{5\pi}{16}$$

[5]

(iv) Evaluate

$$\int_0^{\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta$$

[5]

B9. (i) Evaluate

$$I = \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3}$$

[7]

(ii) (a) Prove the Jordan's lemma. [7]

(b) Using the Jordan's lemma, evaluate

$$I = \int_{-\infty}^{\infty} \frac{x \sin \alpha x}{x^2 + k^2} dx, \quad \alpha > 0, \quad k > 0$$

[6]

END OF QUESTION PAPER