GWANDA STATE UNIVERSITY

CMS2201

FACULTY OF COMPUTATIONAL SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

MATHEMATICAL MODELING

EPOCH MINE CAMPUS

Ms B Kwirira

JUNE 2026: EXAMINATION
Time : 3 hours

Candidates should attempt **ALL** questions from **Section A** (40 marks) and **ANY TWO** questions from **Section B** (30 marks each).

Instruments and Materials

- Calculator.

SECTION A: Answer ALL questions [40].

A1. (a) What is mathematical modeling? [3]

(b) What are the key steps in mathematical modeling process? [5]

(c) Define the term model validation and explain its importance. [5]

A2. A population grows logistically according to the growth model:

$$\frac{dP}{dt} = rP(1 - \frac{P}{K}),$$

where P is the population size, r is the growth rate, and K is the carrying capacity.
Given:
$r = 0,5$ per year
$K = 1000$ individuals
Initial population is 100 individuals.

Find the population size after 5 years. [8]

A3. Write down the mathematical expressions to model the following situations:

(a) The temperature rose from zero to reach a maximum of 20^0C after 5 minutes, before decreasing eventually to zero. [3]

(b) The temperature increased from an initial value of 20^0C to eventually reach 100^0C. [3]

(c.) The temperature varied sinusoidaly between 10^0C and 20^0C with a period of one day, being 10^0C initially. [4]

A4. In the investigation of a homicide or accident death, it is always important to estimate the time of death. From experimental observations it is known that, to an accuracy satisfactory many times, the surface temperature of an object changes at a rate proportional to the difference between the temperature of an object and that of the surrounding environment (the ambient temperature). This is known as the Newton's law of cooling. If $\theta(t)$ is the temperature of an abject at time t, and T is the constant ambient temperature, show that θ satisfy the linear differential equation

$$\frac{d\theta}{dt} = -k(\theta - T).$$

where $k > 0$ is the constant of proportionality. [3]

Suppose that at time $t = 0$ a corpse is discovered to be θ_0. Assuming that at the time of death the body had a normal temperature value of 98.6^0F and that the above differential equation is valid, determine the time of death, t_d. Given that the temperature of the corpse is 85^0 when discovered and 74^0F two hours later, and that the ambient temperature is 65^0F. [6]

SECTION B: Answer ANY two questions [60].

B5. (a) Write down and explain *five* modelling skills that are required when constructing a mathematical model. [10]

(b) A model for growth rate of a population with depensation is given by;

$$g(N) = r(N - K_c)(K - N),$$

where N denotes the population size, r and K are positive constatnts, K_c is a positive or negative constant which is less than K. The differential equation is therefore,

$$\frac{dN}{dt} = f(N) = rN(N - K_c)(K - N)$$

(i.) Find the equilibrium points for $f(N)$ [2]
Sketch the graphs for $f(N)$ and $g(N)$ for the cases
$K_c < -K, -K < K_c < 0$ and $0 < K_c < K$ [6]

(iii.) On the graphs of $f(N)$ indicate the equilibrium points and their nature. [2]

(iv.) Sketch the solution curves of $f(N)$. [2]

(v.) When do we have depensation and when is it critical? [2]

(vi.) Compute the recovery time T_R of the equilibrium $N = K$ with $K_c < K$, [3]

(vii.) Sketch the graph of T_R as a function of K_c, holding all other parameters constant. [2]

(viii.) What happens to T_R as K_c approaches K. [1]

B6. The water supply for the nearby town is provided by a dam situated in a gorge near to the town. The town council would like an estimate of how the depth of water in the dam is likely to change through the dry season, during which time no significant rainfall is expected. The council expects to take water along the pipeline from the dam at an approximately constant rate throught the dry season. In cross section, the floor of the dam rises symmetrically from the centre line at an angle ϕ. Moving away from the damwall, the centre line of the floor slopes upwards at an approximately constant angle θ.

(a.) Show that the volume of water in the dam at a given time is k_1h^3, where h is the maximum depth of water in the dam at that time, find the parameter k_1 in terms of the width of ϕ andθ. [6]

(b.) Assuming that the only loss of water from the dam is through the pipeline, use the conservation of fluid volume to show that a simple model for the rate of change of depth of water in such a dam is

$$\frac{dh}{dt} = -\frac{k_2}{h_2}$$

, where $k-2$ is a parameter.
Hence find an expression for the depth of fluid in the dam in terms of time and parameters of the dam and water supply. [9]

(iii) On one particular dry season, the maximum depth of water at the dam wall as a function of time was as follows;

Depth(m)	35	30	25	20	15
Time(months)	2.1	4.2	6.0	7.5	8.8

By drawing an apppropiate graph, use these data to check the validity of the model developed in (*b.*) [7]

(d.) An alternative model to the depth of water would include loss of water through evapouration. Stating clearly an assumptions you make derive an alternative differential equationfor the rate of change of depth of water. Solve this differential equation and test your model using the data in (*c.*). Which model would you recommend to the council? [8]

B7. (a) Given that $0.342, 0.134, 0.521$ and 0.854 are random numbers from a uniform distribution on [0.1], generate four random numbers from the discrete distribution of a random variable X. [4]

Value(x)	0	1	2	3	4	5
P(X = x)	0.1	0.25	0.2	0.3	0.1	0.05

(b.) In a certain city, the daily consumption of water, X litres is a random variable, but tends to be mainly around $x = 3$. Small and large values of X are very rare. A mathematical modeller decides that a suitable model for the probability density function would be of the form;

$$f(x) = \begin{cases} kxe^{-bx}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

(i) What would be suitable values for parameters k and b? [5]

(ii) Find the cumulative distribution function $F(x)$ and use it to simulate a value X from $RND = 0.445$. [6]

(c.) A firm is carrying out aa cost-cutting exercise and requires your help with an investigation into how it can reduce its transport costs. The firm employs a number of drivers who cover a substantial amount of mileage every day. There has recently been a large increase in their fuel costs and drivers can achieve highe rate of kilometers per litre from their vehicles by driving at a lower speed. This, however increase journey times and the cost of the drivers' time. Given that the vehicles deliver $40km$ per litre at $60km/h$ decreasing steadily with increasing speed to $20km$ per litre at $100km/h$, construct a model which, given relevant information, could give evidence on the optimum driving speed to keep costs to a minimum. [15]