GWANDA STATE UNIVERSITY

CMS 2103

FACULTY OF COMPUTATIONAL SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

LINEAR MATHEMATICS II

EPOCH MINE CAMPUS: FILABUSI

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SEPTEMBER 2024: MAIN EXAMINATION
Time : 3 hours

Candidates should attempt **ALL** questions from **Section A** (40 marks) and **ANY THREE** questions from **Section B** (20 marks each).

Instruments and Materials

- Non-Programmable Calculator.

SECTION A: Answer ALL questions [40].

A1. (a) What is a field? [2]

(b) Write down the matrix notation of the determinant of matrix $A_{n\times n}$. [2]

(c) Define a diagonal matrix. [2]

(d) State the link between an eigenvalue and eigenvector. [4]

A2. Evaluate the determinant of each of the following matrices

(a) $A = \begin{bmatrix} 4 & -5 \\ -1 & -5 \end{bmatrix}$ [2]

(b) $B = \begin{bmatrix} 1 & 3 & -5 \\ 3 & -1 & 2 \\ 1 & -2 & 1 \end{bmatrix}$ [3]

(c) Reduce B to echelon form. [8]

A3. Let $\sigma = 24513$ be permutation in S_5.

(a) Is $\sigma = 24513$ the short way for writing $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix}$ [1]

(b) Find $\sigma \circ \sigma$. [5]

(c) Compute the inverse of σ. [3]

A4. (a) Suppose $T : V \to V$ is linear. Show that each of the following is invariant under T:

(i) $\{0\}$ [2]

(ii) V [2]

(iii) Kernel of T [2]

(iv) Image of T [2]

SECTION B: Answer ANY three questions [60].

B5. (a) Write down the general formula for the minimal polynomial $m(t)$. [2]

(b) Determine all possible (*Hint*: 6) Jordan canonical forms J for a linear operator $T : V \to V$ whose characteristic polynomial

$$\triangle(t) = (t-2)^3(t-5)^2.$$

In each case, find the minimal polynomial $m(t)$. [18]

B6. (a) Define dual bases. [2]

(b) Find the basis $\{\phi_1, \phi_2, \phi_3\}$ that is dual to the following basis of $\mathbb{R}^3$

$$\{v_1 = (1, -1, 3), v_2 = (0, 1, -1), v_3 = (0, 3, -2)\}$$

The linear functionals may be expressed in the form

$$\begin{aligned} \phi_1(x, y, z) &= a_1x + a_2y + a_3z \\ \phi_2(x, y, z) &= b_1x + b_2y + b_3z \\ \phi_3(x, y, z) &= c_1x + c_2y + c_3z \end{aligned}$$

[16]

(c) Comment on your answer in (b). [2]

B7. (a) Let ϕ be the linear functional on $\mathbb{R}^2$ defined by $\phi(x, y) = x - 2y$. For each of the following linear operators T on $\mathbb{R}^2$, find $(T^t(\phi))(x, y)$:

(i) $T(x, y) = T(x, 0)$ [2]

(ii) $T(x, y) = T(y, x + y)$ [2]

(b) Apply change of variables on $f(x, y) = x^2 + xy + y^2$ by determining the

(i) Matrix A representation of f. [2]

(ii) Eigenvalues and eigenvectors of the matrix A. [4]

(iii) Normalize the eigenvectors to obtain an orthonormal basis. [2]

(iv) Construct the change of basis matrix P using the normalized eigenvectors. [2]

(v) Diagonalize the matrix A using the change of basis matrix P. [4]

(vi) Express the function $f(x, y)$ in the new coordinate system using the diagonal matrix D. [2]

B8. (a) Give the condition for a matrix to be symmetric. [2]

(b) Determine whether the following matrix is symmetric, Hermitian and normal.

$$\begin{bmatrix} 3 & 2-i & 4+i \\ 2-i & 6 & i \\ 4+i & i & 7 \end{bmatrix}$$

[8]

(c) Consider the following diagonal matrices A, E_1, E_2, E_3

$$A = \begin{bmatrix} 2 & & & \\ & 3 & & \\ & & 3 & \\ & & & 5 \end{bmatrix}, E_1 = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{bmatrix}$$

Verify the conditions

(i) $A = \lambda_1 E_1 + \lambda_2 E_2 + \cdots + \lambda_r E_r$ [2]

(ii) $E_1 + E_2 + \cdots + E_r = I$ [2]

(iii) $E_1^2 = E_1, E_2^2 = E_2, \cdots, E_r^2 = E_r$ [3]

(iv) $E_i E_j = 0$ for $i \neq j$. [3]

END OF QUESTION PAPER