GWANDA STATE UNIVERSITY

CMS1203

FACULTY OF COMPUTATIONAL SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

ANALYSIS 1

EPOCH MINE CAMPUS

Ms B Kwirira

JUNE 2025: EXAMINATION
Time : 3 hours

Candidates should attempt **ALL** questions from **Section A** (40 marks) and **ANY TWO** questions from **Section B** (30 marks each).

Instruments and Materials

- Calculator.

SECTION A: Answer ALL questions [40].

A1. By stating a standard result, prove that between any two real numbers there is an irrational number. [6]

A2. Let $b \in \mathbb{R} : \; b > 0$. Prove that $\exists a \in \mathbb{R} : \; a^2 = b$. [10]

A3. Let A and B be bounded non empty subsets of $\mathbb{R}$ where $A+B = \{a+b : \; a \in A, b \in B\}$. Prove that $\sup(A+B) = \sup A + \sup B$. [6]

A4. Define the notion of a Cauchy sequence in $\mathbb{R}$. By assuming standard results, prove that a convergent sequence in $\mathbb{R}$ is Cauchy. [2,6]

A5. Let

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if} \quad x \neq 0 \\ 0 \quad \text{if} & x = 0 \end{cases}$$

show that f is not differentiable at the origin even though it is continuous at the origin. [6]

A6. Let $f(x) = x^3$ be a function defined on a closed bounded interval $[0,1]$. Show that $f(x)$ is Riemann integrable on $[0,1]$. [4]

SECTION B (60 marks)

Candidates may attempt THREE questions being careful to number them B? to B?.

B7. (a) Let $a_1 = 1, a_{n+1} = \dfrac{1}{4 + a_n^2}, \quad n \geq 1$

(i) Show that $\dfrac{1}{5} < a_n \leq \dfrac{1}{4}, \ \forall n > 1$. [3]

(ii) Prove that $| a_{n+1} - a_n | \leq c | a_n - a_{n-1} |$ for some $c : 0 < c < 1$. [4]

(iii) Show that $\{a_n\}$ is a Cauchy sequence and deduce that $\{a_n\}$ converges to a fixed point of the function $f(x) = \dfrac{1}{4}(1 - x^3)$. [8]

(b) State the Nested Interval Theorem and verify it for the sequence $I_n = \left\{ \left[0, \dfrac{1}{n} \right] \right\}$. [5]

(c) Let S be a bounded infinite set of real numbers. By assuming standard results, prove that S has at least one limit point. [10]

B8. (a) Prove that the only numbers in $\mathbb{N}$ with square roots in $\mathbb{Q}$ are the square numbers. [9]

(b) Let $a, b \in \mathbb{R}$. Prove that if $a \leq b + \dfrac{1}{n}, \ \forall n \in \mathbb{N}$ then $a \leq b$ [you may assume standard results]. [6]

(c) Let A be a non empty subset of real numbers which is bounded above. Prove that a real number a is a supremum of the set A if and only if

(i) a is an upper bound of A and

(ii) $\forall \epsilon > 0, \ \exists a_\epsilon \in A : \ a_\epsilon > a - \epsilon$. [9]

(d) Let $a, b, c \in \mathbb{R}$. Prove that if $a + b = a + c$ then $b = c$. Hence or otherwise prove that $\forall a \in \mathbb{R}, \ a \cdot 0 = 0$. [6]

B9. (a) Let f be a continuous function on a closed and bounded interval domain $[a, b]$ and differentiable in an open interval (a, b) with $f(a) = f(b)$. Prove that $\exists \xi \in (a, b) : f'(\xi) = 0$. [10]

(b) State and prove the **generalised mean-value theorem** [you may assume standard results]. [12]

(c) Let f be a continuous function on a closed and bounded interval domain $[a, b]$. Prove that $f(x)$ is bounded on $[a, b]$ [you may assume standard results]. [8]

B10. (a) Prove that if $f(x)$ is a monotone function on a closed bounded interval $[a, b]$ then $f(x)$ is Riemann integrable. [10]

(b) Let $f(x)$ be a continuous function on $[a, b]$. Prove that there exists $\xi \in (a, b)$: $\int_a^b f(x)dx = f(\xi)(b - a)$. [10]

(c) Let $f(x)$ be a bounded integrable function on a closed bounded interval $[a, b]$ and $F(x) = \int_a^x f(t)dt \ \forall x \in (a, b)$. Prove that $F(x)$ is continuous on $[a, b]$. [10]