# GWANDA STATE UNIVERSITY

## FACULTY OF COMPUTATIONAL SCIENCES

### DEPARTMENTS OF MATHEMATICS AND STATISTICS

**PROBABILITY I**

**CMS 1103**

**Examination Paper**

**NOVEMBER 2024**

**Time Allowed:** **3 hours**

**Total Marks:** **100**

**Examiner's Name:** **Mr. M. Mpofu**

**INSTRUCTIONS**

Candidates should answer **ALL** questions in Section A and attempt **ANY THREE** questions in Section B.

**ADDITIONAL REQUIREMENTS**

Scientific calculator

## SECTION A (40 marks)
**Answer ALL questions from this section.**

**A1.** (i) Suppose that the probability of event $A$ is $P(A) = 0.6$, and the probability of event $B$ is $P(B) = 0.5$. If $P(A \cap B) = 0.3$, find $P(A \mid B)$ and $P(B \mid A)$. [4]

(ii) For any events $A$ and $B$, if $A \subseteq B$, prove $P(A) \leq P(B)$. [4]

**A2.** State and prove Chebyshev's Inequality [6]

**A3.** (i) A fair die is tossed. Let $X$ denote twice the number appearing, and let $Y$ denote 1 or 3 according as an odd or an even number appears. Find the distribution, expectation, and variance of

(a) $X$ [6]

(b) $X + Y$ [6]

(ii) Prove that $Var[X + k] = Var[X]$ [4]

**A4.** (i) Let $X$ and $Y$ have the joint density $f(x, y) = \frac{1}{\pi}$, $x^2 + y^2 \leq 1$. Find the conditional distribution of $Y$ given that $X = x$. [5]

(ii) Let $X$ and $Y$ be two continuous random variables with joint probability distribution $f(x, y) = 4xy$, $0 < x < 1$, $0 < y < 1$ and zero elsewhere. Find the joint probability distribution $W = X^2$ and $Z = XY$. [5]

## SECTION B (60 marks)
**Answer ANY TWO questions from this section.**

**B5.** (i) Define

(a) Probability space [2]

(b) Exhaustive events [2]

(ii) Two football teams $M$ and $C$ each have one game left to play (not against each other) in the season. If $M$ wins and $C$ does not win, or if $M$ draws and $C$ loses, then $M$ wins the championship. Otherwise $C$ wins the championship. The probabilities that $M$ wins, draws or loses the last game are $\frac{1}{2}$, $\frac{1}{6}$, and $\frac{1}{6}$, respectively. The probabilities that $C$ wins, draws or loses the last game are $\frac{2}{3}$, $\frac{1}{6}$, and $\frac{1}{6}$, respectively. What is the probability that

(a) $M$ wins the championship? [4]

(b) $C$ has drawn the last game given that $M$ has won the championship? [5]

(iii) A diagnostic test for a disease gives a positive result with probability 0.98 for people who have the disease, and a negative result with probability 0.99 for people who do not have the disease. Suppose 3% of the population have the disease.

(a) A person is selected at random from the population and given the test. If the result is positive, what is the probability that this person has the disease? [3]

(b) Suppose a person, initially selected at random from the population, is given the test once and the result is positive. This person is then given the test, independently, a second time and the result is again positive. What is the probability that this person has the disease? [4]

**B6.** (i) Let $X$ be a continuous random variable with the probability density function:

$$f(x) = \begin{cases} \frac{1}{2}e^{-|x|} & \text{for } x \in \mathbb{R} \\ 0 & \text{otherwise} \end{cases}$$

Find the mean, and variance of $X$. [8]

(ii) Let $c$ be a constant. Show that $Var(cX) = c^2 Var(X)$. [5]

(iii) Let $X$ be a random variable with the binomial distribution $b(k;\ n,\ p)$. Prove

(a) $E[X] = np$ [3]

(b) $Var[X] = npq$ [4]

**B7.** Consider the following joint density function for random variables $X$ and $Y$

$$f(x) = \begin{cases} k(1-y), & \text{if } \ 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

(i) Find the value of $k$, [3]

(ii) Evaluate $E[X]$ and $E\left[X^2\right]$, and hence evaluate $Var[x]$. [5]

(iii) Derive the conditional density, $f_{Y|X}(y)$ and the conditional expectation $E[1 - y]/X$. Hence or otherwise, evaluate $E[Y]$ and $Cov(X, Y)$. [8]

(iv) Evaluate $P(Y < 2x)$. [4]

**B8.** The random variable $Y$ has probability density function

$$f(y) = k(y + y^3), \quad 0 < y < 2,$$

and zero otherwise, where $k$ is a positive constant.

(i) Show that $k = \frac{1}{6}$. [4]

(ii) Show that the cumulative distribution function is

$$F(y) = \begin{cases} 0, & \text{if} y \leq 0, \\ \left(\frac{y^2}{12}\right) + \left(\frac{y^2+2}{2}\right), & \text{if} 0 < y < 2, \\ 1, & \text{if} y \geq 2 \end{cases}$$

Hence find $P\left(\frac{1}{2} \leq Y \leq \frac{3}{2}\right)$. [8]

(iii) Find the variance of $Y$. [8]

**END OF QUESTION PAPER**