GWANDA STATE UNIVERSITY

CMS 2204

FACULTY OF COMPUTATIONAL SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

General Topology

EPOCH MINE CAMPUS: FILABUSI

Mr M NDLOVU

JUNE 2025: EXAMINATION
Time : 3 hours

Candidates should attempt **ANY FOUR** questions from this paper (25 marks each).

Instruments and Materials

- Non-Programmable Calculator.

A1. Question 1: Fundamental Topological Concepts [25]

(a) Prove that the interval $(2, 4)$ is homeomorphic to $(-3, -1)$ by constructing an explicit homeomorphism. [8]

(b) Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Determine whether:

i. X is connected [4]

ii. X is Hausdorff [4]

(c) Prove that any continuous function $f : \mathbb{R} \to \mathbb{Q}$ (where $\mathbb{Q}$ has subspace topology) must be constant. [9]

A2. Question 2: Continuity and Homeomorphisms [25]

(a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the multiplication map $f(x, y) = xy$.
Prove that the preimage $f^{-1}((1, 2))$ is open in $\mathbb{R}^2$. [10]

(b) Construct an explicit homeomorphism between:

- The unit cube $[0, 1]^3 \subset \mathbb{R}^3$ and
- The closed unit ball $\{(x, y, z) | x^2 + y^2 + z^2 \leq 1\}$ [15]

A3. Question 3: Product and Quotient Spaces [25]

(a) Prove that if S and T are Hausdorff spaces, then $S \times T$ is Hausdorff. [10]

(b) Let $\sim$ be the equivalence relation on $S^1 \times [0, 1]$ that:

- Identifies all points with $t = t' = 1$
- Identifies all points with $t = t' = 0$
- Leaves other points equivalent only to themselves $((x, y), t) = ((x', y'), t')$

Prove that $S^1 \times [0, 1]/\sim$ is homeomorphic to S^2. [15]

A4. Question 4: Connectedness and Compactness [25]

(a) Prove that $\mathbb{R}^n$ with the standard topology is connected. [10]

(b) Investigate whether $[0, \frac{1}{2})$ is compact. Either:

- Provide a proof of compactness, or
- Exhibit an unbounded continuous function $f : [0, \frac{1}{2}) \to \mathbb{R}$ [15]

A5. Question 5: Special Topological Spaces [25]

(a) Consider $\mathbb{Z}$ with:

(i) The standard topology [5]

(ii) The indiscrete topology [5]

For each case, determine whether the space is connected, compact, and Hausdorff.

(b) Verify that the collection of open sets defined for the real line with a double point forms a valid topology. [15]

A6. Question 6: Advanced Constructions [25]

(a) Derive the formulas for stereographic projection:

i. From $S^1 - \{(0,1)\}$ to $\mathbb{R}$ [6]

ii. From $S^2 - \{(0,0,1)\}$ to $\mathbb{R}^2$ [6]

(b) Let B be a collection of subsets of T satisfying:

i. Every element of T belongs to at least one set in B

ii. B is closed under finite intersections

Prove that the collection of all unions of sets from B forms a topology on T. [13]

Total Marks: 100
(Answer any four questions, 25 marks each)