GWANDA STATE UNIVERSITY

FACULTY OF COMPUTATIONAL SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

Algebra I

CMS 2203

Semester Two 2025/2026 : Examination

This examination paper consists of 2 printed pages, exclusive of this cover page.

Date:	**MAY 2026**
Total:	**100 Marks**
Time:	**3 hours**
Examiner's Name:	**Mr. M. Ndlovu**

INSTRUCTIONS

Candidates should attempt **ALL** questions from **Section A** which carries **40 marks**. Answer **ANY THREE** questions from **Section B** each question carries **20 marks**. **Section B** carries **60 marks**.

- All answers and all rough work must be written in the answer booklet(s) provided.
- A line should be drawn through any rough work to indicate to the examiner that it is not part of the work to be graded.
- Black and blue pen only must be used for written answers and pencil for all drawings and sketches.

ADDITIONAL MATERIALS

- Calculator (non-programmable)
- Pens and Pencils

SECTION A: Answer ALL questions [40].

A1. (a) Note that, $(\mathbb{Z}, +)$ is a group, but $(\mathbb{N}, +)$ is not.
Which group axiom fails for $(\mathbb{N}, +)$? **(2 Marks)**

(b) In the dihedral group D_4 (symmetries of a square),
list the eight symmetries. **(3 Marks)**

(c) Prove that if n is odd then n^2 is odd. **(3 Marks)**

(d) Recall that $7 \mid 91$ because $91 = 7 \times 13$.
Check whether $4 \mid 26$. If not, explain why. **(2 Marks)**

(e) Consider the set $3\mathbb{Z} = \{3k \mid k \in \mathbb{Z}\}$.
Prove, that $3\mathbb{Z}$ is closed under addition. **(4 Marks)**

A2. (a) In $\mathbb{Z}_{12}$, list all elements of the subgroup generated by 9. **(3 Marks)**

(b) Find the multiplicative inverse of 3 in $\mathbb{Z}_7$. **(3 Marks)**

(c) In the group U_{35}, determine whether 4 belongs to U_{35}.
Justify your answer. **(3 Marks)**

A3. (a) Given a non-property: $a \mid bc$ does not imply $a \mid b$ or $a \mid c$.

(i) Provide the counterexample given: $4 \mid 6 \times 2$ but $4 \nmid 6$ and $4 \nmid 2$. **(2 Marks)**

(ii) Now give a different counterexample with numbers of
your choice. **(2 Marks)**

(b) For $C = \{n \in \mathbb{Z} \mid n \in 2\mathbb{Z} \text{ and } n \in 3\mathbb{Z}\}$.
It is conjectured that $C = 6\mathbb{Z}$.
Verify that -12 belongs to both $2\mathbb{Z}$ and $3\mathbb{Z}$. **(3 Marks)**

(c) The center of D_4 is given as $Z(D_4) = \{\epsilon, r_{180}\}$.
Check whether r_{90} commutes with the horizontal reflection h. **(3 Marks)**

A4. (a) List the generators of $\mathbb{Z}_{12}$. **(2 Marks)**

(b) State the First Isomorphism Theorem. **(2 Marks)**

(c) It can be shown that $\gcd(5, 8) = 1$ because $5(-3) + 8(2) = 1$.
Find integers x, y such that $6x + 11y = 1$. **(3 Marks)**

SECTION B: Answer ANY three questions [60].

B5. Consider the group $(\mathbb{Z}_{10}, +)$.

(a) Write the addition table for the subset $H = \{0, 2, 4, 6, 8\}$. **(4 Marks)**

(b) Show that H is a subgroup of $\mathbb{Z}_{10}$. **(4 Marks)**

(c) Find all left cosets of H in $\mathbb{Z}_{10}$. **(4 Marks)**

(d) Determine the index $[\mathbb{Z}_{10} : H]$. **(2 Marks)**

B6. In the dihedral group D_4:

(a) Write the elements of the centralizer $C(h)$ for the horizontal reflection h. **(4 Marks)**

(b) Prove that $C(h)$ is a subgroup of D_4. **(6 Marks)**

(c) Show that the left coset $r_{90}C(h)$ is not equal to the right coset $C(h)r_{90}$. **(6 Marks)**

(d) What does this tell you about the normality of $C(h)$? **(4 Marks)**

B7. Let $U_{13} = \{1, 2, \ldots, 12\}$ under multiplication modulo 13.

(a) Show that 2 is a generator of U_{13} by computing its powers until you reach 1. **(6 Marks)**

(b) Find the order of the element 5. **(4 Marks)**

(c) Let $H = \{1, 3, 9\}$. Verify that H is a subgroup of U_{13}. **(4 Marks)**

(d) Construct the quotient group U_{13}/H and write its multiplication table. **(6 Marks)**

B8. Consider the matrix group $S(\mathbb{Z}_{10})$ (Special Linear Group).

(a) Verify that the matrix $\begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$ has determinant 1 in $\mathbb{Z}_{10}$. **(3 Marks)**

(b) Show that the product $\begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (computed in $\mathbb{Z}_{10}$) also lies in $S(\mathbb{Z}_{10})$. **(5 Marks)**

(c) Prove that the set $S(\mathbb{Z}_{10})$ is closed under matrix multiplication. (Hint: use $\det(AB) = \det A \det B$.) **(6 Marks)**

(d) Find the inverse of $\begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$ in $S(\mathbb{Z}_{10})$. **(6 Marks)**