GWANDA STATE UNIVERSITY

CMS 2203

FACULTY OF COMPUTATIONAL SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

Algebra I

EPOCH MINE CAMPUS: FILABUSI

Mr M NDLOVU

JUNE 2025: SPECIAL EXAMINATION
Time : 2.5 hours

Candidates should attempt **ANY FOUR** questions from this paper (25 marks each).

Instruments and Materials

- Non-Programmable Calculator.
- Graph Paper (provided).

A1. Question 1: Fundamental Concepts [25]

(a) Define the following terms:

(i) Group [2]

(ii) Normal subgroup [2]

(iii) Group homomorphism [2]

(b) Explain why the set of odd integers under addition is not a group, while the set of even integers is. [5]

(c) Show that the function $f : (\mathbb{R}^*, \times) \to (\mathbb{R}^*, \times)$ defined by $f(x) = |x|$ is a group homomorphism. [6]

(d) Compare and contrast cyclic groups and dihedral groups, giving one example of each. [8]

A2. Question 2: Group Construction and Analysis [25]

Let $G = \{1, -1, i, -i\}$ under complex multiplication.

(a) State the Cayley table for G. [5]

(b) Explain why G is isomorphic to $\mathbb{Z}_4$. [6]

(c) Find all proper subgroups of G and determine which are normal. [7]

(d) Assess whether G can be expressed as a direct product of smaller groups. Justify your answer. [7]

A3. Question 3: Homomorphisms and Quotient Groups [25]

Consider the group $\mathbb{Z}_{24}$ under addition modulo 24.

(a) List all generators of $\mathbb{Z}_{24}$. [5]

(b) Explain why the function $\phi : \mathbb{Z}_{24} \to \mathbb{Z}_{12}$ defined by

$$\phi(x) = 3x \mod 12$$

is a homomorphism. [6]

(c) Find $\ker(\phi)$ and $\text{im}(\phi)$ for the homomorphism above. [7]

(d) Construct an isomorphism between $\mathbb{Z}_{24}/\langle 8 \rangle$ and another cyclic group. [7]

A4. Question 4: Advanced Group Theory [25]

Let S_3 be the symmetric group on 3 elements.

(a) Write all elements of S_3 in cycle notation. [5]

(b) Explain why $A_3 = \{e, (123), (132)\}$ (the alternating subgroup) is normal in S_3. [6]

(c) Compute the conjugacy classes of S_3. [7]

(d) Compare the structure of S_3 with that of the dihedral group D_3. Are they isomorphic? [7]

A5. Question 5: Real-World Applications [25]

(a) State Lagrange's Theorem and give its formula. [5]

(b) Explain how group theory applies to solving the Rubik's cube. [6]

(c) Using group theory concepts, determine how many distinct ways a necklace with 5 differently colored beads can be arranged if rotations are considered identical. [7]

(d) Design a problem about symmetry groups in chemistry (molecular symmetry) and solve it. [7]

A6. Question 6: Proof and Theory [25]

(a) State the First Isomorphism Theorem for groups. [5]

(b) Explain the proof that every cyclic group is abelian. [6]

(c) Prove that the center $Z(G)$ of any group G is a normal subgroup. [7]

(d) Critique the following statement:

"All groups of order 4 are isomorphic to each other."

[7]

Total Marks: 100
(Answer any four questions, 25 marks each)