GWANDA STATE UNIVERSITY

CMS2201

FACULTY OF COMPUTATIONAL SCIENCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

MATHEMATICAL MODELLING

EPOCH MINE CAMPUS

Ms B Kwirira

JUNE 2025: EXAMINATION
Time : 3 hours

Candidates should attempt **ALL** questions from **Section A** (40 marks) and **ANY TWO** questions from **Section B** (30 marks each).

Instruments and Materials

- Calculator.

SECTION A: Answer ALL questions [40].

A1. Write down the major factors that must be considered when constructing a mathematical model. [3]

A2. Rival supermarkets enter into a price war for dog food. Supermarket A drops its price and as a resultthe daily sales volume of supermarket B is changed by an amount proportional to;

(a.) The difference between B's price and A's price.

(b.) The difference between$B$'s price and the recommended retail price.

(c.) The differential between A's price and the recommended retail price

Construct a model for the new sales volume of supermarket B. [7]

A3. Each batch of a chemical used in drug manufacture is tested for impurities. The percentage of impurity is X. A mathematical model decides that a suitable model for the probability density function would be given by:

$$f(x) = \{ \begin{array}{ll} kx, & 0 < x \leq 1 \\ \frac{1}{3}k(4-x), & 1 < x \leq 4 \\ 0, & otherwise \end{array}$$

, where k is a constant.

(i) Find the suitable value of k. [3]

(ii) Find the cumulative distribution function,$F(x)$. [5]

(iii) Simulate the percentage of impurity if ;

(i) $RND = 0.2$,

(ii) $RND = 0.5$. [5]

A4. At a school with N pupils, it was found out that 10 pupils were infected with smallpox. It was assumed that the rate of change in the infected pupils is proportional to the product of the number of pupil who have the disease with the number that are disease free.

(a.) By letting $I(t)$ denote the number of pupils with smallpox at time t, show that

$$\frac{dI}{dt} = kI(N - I)$$

, where k is a positive constant. [3]

(b.) Find the general solution of the equation in $(a.)$ [5]

(c.) Show that as $t \to \infty$ the infected Population I converges to N [3]

(d.) Given that the school had 1000 pupils, how long will it take for a quarter of the pupils to contract the disease, given that after five weeks 15 pupils had contracted the disease? [6]

SECTION B: Answer ANY two questions [60].

B5. The logistic growth equation for the natural growth rate of a populatrion is

$$\frac{dN}{dt} = rN(K - N),$$

where $N(t)$ is the population size at time t and K is the carrying capacity of the population.

(a.) With the aid of a graph of $\frac{dN}{dt}$, show that the population converge to the carrying capacity, K as $t \to \infty$ [3]
The above population is the subjected to harvesting by an amount denoted by $h(N)$. The model is then modelled by;

$$\frac{dN}{dt} = rN(K - N) - E(N),$$

where $f(N) = rN(K - N)$ and $h(N) = EN$ are natural growth rate of the population and the rate of reduction of the population respectively.

(b.) find the equilibrium points of;

$$\frac{dN}{dt} = rN(K - n) - EN$$

Hence, sketch the graph of

$$\frac{dN}{dt} = rN(K - n) - EN$$

as a function of N for cases $E < rK$ and $E > rK$ showing the nature of the equilibrium point. [7]

(c.) Sketch the graph of the equilibrium point as a function of E, showing on the grapoh the nature of the population when $E < rK$ and $E > rK$
What happens to the population if E is increased above rK. [5]

(d.) Write down the birfurcation point. [2]

(e.) Show that a strictly positive sustainable yield is only posible if $E < rK$. [3]

(f.) Write down the value of E that maximises the maximum sustainable yield. Hence write down the sustainable yield. [4]

(g.) Show that when the maximum sustainable yield is maximum, the stable population is half the carrying capacity. [3]

(h.) Sketch a graph of the maximum sustainable yield as a function of E, showing the maximum sustainable yield and where it occurs. [3]

B6. (a) Consider a chemical which can diffuse between three identical compartments. The chemical flows from one compartment to an adjacent one at a rate which is proportional to the difference in the concentrations in the compartments and towards the compartment with lower concentration. In addition the chemical leaks out of each compartment at a rate proportional to its concentration in the compartment. If the concentrations of the chemical in the three compartments are c_1, c_2 and c_3 the equations for this system are;

$$\begin{aligned} \dot{c_1} &= -\alpha(c_1 - c_2) - \gamma c_1 \\ \dot{c_2} &= \alpha(c_1 - c_2) - \alpha(c_2 - c_3) - \gamma c_2 \\ \dot{c_3} &= \alpha(c_2 - c_3) - \gamma c_3 \end{aligned}$$

where α and γ are strictly positive diffusion constants.
Draw a compartmentalised diagram for the above system. [8]

(b) The Bulawayo water supply is provided by a series of dams situated at various points around the city. The water flows under the force of gravity, along the pipelines from the dams to the central reservoir on the edge of the town.

The town council would like an estimate of the depth of the water in the dam is likely to change throughout the dry season, during which time no significant rainfall is expected. There is very little change in altitude between the start of any pipeline in the dam and its finish at the central reservoir.
The dams are sited in the valley with steep sides and have a rectangular shape. All the sides of the dams and the dam walls can be considered vertical.

(i) List at least six factors which you think maybe relevant in a model for the depth of a fluid in such a dam. [5]

(ii) Write down the dimensions of each of the factors identified above and categorize the factors as constant, parameter, input or output variable. [4]

(iii) It is given, from the conservation of water volume, that a simple model for the rate of change of depth of fluid in a dam is;

$$\frac{dh}{dt} = -\lambda v,$$

where v is the average velocity of water leaving the pipeline. Assumming that v, is only a function of depth of water in the dam, the density of the fluid and the acceleration due to gravity, use dimensional analysis to find an expression for v. Hence find an expression for the depth of water in the dam as a function of time. [6]

(d.) In one particular dry season, the depth of water in one of the dams as a function of time was as follows.By drawing an appropiate graph, use these data to check the validity of the model developed above. [7]

B7. The following table gives the estimated number of fish y in a lake over a number of years.

Time(years)	0	1	2	3	4	5	6
y(x10^3	500	700	820	890	930	960	980

(a.) Show that y can be reasonably represented by a logistic model

$$\frac{L}{y} - 1 = (\frac{L}{y_0} - 1)e^{kt},$$

and obtain estimates for parameters L and k. [15]

(b.) Compare the model's prediction with the actual data. [8]

(c.) Write down a constant, a parameter, an input variable and an output variable that you think maybe significant in modelling the flow of a viscous liquid under gravity through a pipe. [7]