GWANDA STATE UNIVERSITY

FACULTY OF COMPUTATIONAL SCIENCES

ANALYSIS 1

SMS 1113

This examination paper consists of 5 pages

Date:	**SEPTEMBER 2023**
Total Marks:	**100**
Time:	**2 hours**
Examiner's Name:	**Ms B. Kwirira**

<u>INSTRUCTIONS</u>

This paper consists of six questions. Answer all questions in section A and answer any TWO questions in section B.

Use of calculator is permissible

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SECTION A: Answer ALL questions [40].

A1. (a) By stating standard results, prove that the set of transcendental numbers $\mathbb{T}$ is uncountable. [6]

(b) Let F be a field. Prove that the additive inverse of F is unique. [6]

(c) Use the intermediate value theorem and Rolle's theorem to show that the equation $x^5 + 2x^3 + x - 3 = 0$ has exactly one real root. [8]

A2. (a) Prove that every Cauchy sequence of real numbers is convergent. [6]

(b) Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if} \quad x \neq 0 \\ 0 \quad \text{if} & x = 0 \end{cases}$$

show that f is both continuous and differentiable at the origin. [8]

(c) Let $f(x) = x^2$ be a function defined on a closed bounded interval $[0, 1]$. Show that $f(x)$ is Riemann intergrable on $[0, 1]$. [6]

SECTION B (60 marks)

Candidates may attempt THREE questions being careful to number them B? to B?.

B3. (a) Let $a_1 = 1$, $a_{n+1} = \dfrac{2}{3 + a_n^2}$.

(i) Show that $\dfrac{1}{2} \leq a_n < 1$ and $| a_{n+1} - a_n | < \dfrac{4}{9} | a_n - a_{n-1} |, \forall n > 1$. [6]

(ii) Show that this sequence is Cauchy and deduce that it converges to a fixed point of the function $f(x) = \dfrac{1}{3}(2 - x^3)$. [9]

(b) State the Principle of Monotone Bounded Convergent sequences and prove for monotone increasing sequences only. [10]

(c) Verify the Nested Interval Theorem for the sequence $I_n = \left\{ \left[0, \dfrac{1}{n} \right] \right\}$. [5]

B4. (a) Let f be a continuous function on a closed and bounded interval domain $[a, b]$. Prove that $f(x)$ is uniformly continuous on $[a, b]$ [Hint: you may assume some standard results]. [8]

(b) Read the sketch proof of Rolle's theorem and answer questions that follow.
Sketch Proof: If $f(x)$ is a constant on $[a, b]$ then $f'(\xi) = 0, \ \forall \xi \in (a, b)$. Now, consider $f(x)$ which is not a constant on $[a, b]$, it follows that $f(x)$ is bounded on $[a, b]$ and it attains its minimum and maximum values m and M respectively on $[a, b]$. Moreover $m \neq M$. If $M \neq f(a) = f(b)$, then $f(a)$ and $f(b)$ are less than M. Since M is the maximum, there exists a point $\xi \in (a, b) : f(\xi) = M$. Since $f(\xi)$ is the least upper bound of the set $f[a, b]$, it implies that
$f(x) \leq f(\xi) = M, \ \forall x \in [a, b]$. Hence $f'(\xi) = 0$.

(i) State Rolle's theorem. [2]

(ii) Why is $f(x)$ a bounded function on $[a, b]$? [Hint: you may quote standard results]. [2]

(iii) Write down the standard result that has been used to reach the argument that f attains its minimum and maximum values. [2]

(iv) 'If $M \neq f(a) = f(b)$ then $f(a)$ and $f(b)$ are less than M.' Justify this statement. [2]

(v) What conclusion can you draw from the fact $f(x) \leq f(\xi) = M, \forall x \in [a, b]$? [2]

(c) State the **generalised mean value theorem**. Hence or otherwise prove the following theorem:
Theorem. Let f and g be functions that are continuous at point a and differentiable in (a, b) with $g'(x) \neq 0$ in (a, b). If $\lim_{x \to a^+} f(x) = 0$, $\lim_{x \to a^+} g(x) = 0$ and $\lim_{x \to a^+} \dfrac{f'(x)}{g'(x)} = l$ then $\lim_{x \to a^+} \dfrac{f(x)}{g(x)} = l$. [12]

B5. (a) Let A be a non empty subset of real numbers which is bounded above. Prove that a real number a is a supremum of the set A if and only if

(i) a is an upper bound of A and

(ii) $\forall \epsilon > 0, \ \exists a_\epsilon \in A : \ a_\epsilon > a - \epsilon.$ [10]

(b) It is well known in elementary real analysis that, "$\forall b \in \mathbb{R}^+, \ \exists! a \in \mathbb{R}^+ : \ a^2 = b$". Read the following sketch proof of this result and answer the questions that follow.

Sketch Proof: Define $L = \{x \in \mathbb{R} : \ 0 \leq x \text{ and } x^2 < b\}$ and $M = \{x \in \mathbb{R} : \ 0 \leq x \text{ and } x^2 > b\}$. L and M are non empty sets. Now, $\forall l \in L, \ \forall m \in M$ we have $0 \leq l^2 < b < m^2$. So $l^2 < m^2$ and we have $l < m$. Hence $\exists a : \ a = \sup L$. Now $\forall l \in L, \ n > \dfrac{2l+1}{b-l^2}$, implies that $(l + \dfrac{1}{n})^2 < b$. Hence $a^2 \geq b$. Similarly, by using the argument that $\forall m \in M, \ n > \dfrac{2m+1}{m^2-b}$, implies that $(m - \dfrac{1}{n})^2 > b$, we can conclude that $a^2 \leq b$. So $a^2 = b$.

(i) Justify the fact that L and M are non empty sets. [2]

(ii) Which property has been used to conclude that $l^2 < m^2$? Further, what arguments can we use to conclude that $l < m$? [2]

(iii) What conclusion can you make from the fact that $\forall l \in L, \ \forall m \in M, \ l < m$? [2].

(iv) State and write down the axiom that has been used to reach the fact that $\exists a : \ a = \sup L$. [2]

(v) What conclusion can you draw from the fact that $\forall l \in L, \ (l + \dfrac{1}{n})^2 < b$? [2]

(vi) State the property that has been used to come to a conclusion that $a^2 = b$. [2]

(c) State and prove the Rational Density Theorem. [8]

B6. (a) Prove that a bounded $f(x)$ is Riemann integrable on $[a, b]$ if and only if given $\epsilon > 0$ there is a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$. [10]

(b) It is well known in elementary analysis that "if f and g are real valued functions on a closed and bounded interval $[a, b]$ with $g'(x) \geq 0$ then there exists $c \in [a, b]$: $\displaystyle\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$". Read the sketch proof of the aforementioned theorem and answer questions that follow.

Sketch Proof: If $g(x) = 0$ then the proof is trivial. So lets consider $g(x) > 0$, it follows that $f(x)$ attains its minimum and maximum values m and M respectively such that $m \leq f(x) \leq M$. So $mg(x) \leq f(x)g(x) \leq Mg(x)$. Now $\int_a^b mg(x)dx \leq \int_a^b f(x)g(x)dx \leq \int_a^b Mg(x)dx$

$\Longrightarrow m\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M\int_a^b g(x)dx.$

Hence $m \leq \dfrac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \leq M.$

(i) State standard results that has been used to reach the conclusion that there exists m and M such that $m \leq f(x) \leq M$. [2]

(ii) Which condition in the theorem has been used to reach the fact that $mg(x) \leq f(x)g(x) \leq Mg(x)$? [2]

(iii) What conclusion can you draw from the argument that $m \leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \leq M$? [2]

(iv) Following your answer to item (iii), why does there exists $c \in [a, b] : \int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx$? [2]

(c) State and prove the **fundamental theorem of integral calculus**. [12]

END OF QUESTION PAPER